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COMMENT

Comment on transport processes on fractal structures

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Abstract. The scaling exponents for the 'depleted Sierpinski gasket' derived by Havlin and Ben-Avraham are shown to differ sharply from those derived by the method of Given and Mandelbrot. The discrepancy is analysed and it is shown that the argument used by Havlin and Ben-Avraham is incorrect.

A recent paper, Havlin and Ben-Avraham (1983), to be referred to as HBA, derives the scaling exponents for a certain fractal lattice one may call 'depleted Sierpinski gasket' (figure 2) whose lattice approximants converge asymptotically to the full Sierpinski gasket (figure 1), but have lower symmetry. However, by applying to the same lattice the methods used in Given and Mandelbrot (1983), to be referred to as GM, one obtains sharply different exponents. This comment analyses the discrepancy, and shows that the argument used by HBA is incorrect. It assumes that in this depleted lattice, the scaling behaviour is attained immediately, whereas in fact it is only attained asymptotically and extremely slowly. This error deserves exploration because it touches upon a conceptual point, and risks being repeated in other contexts.

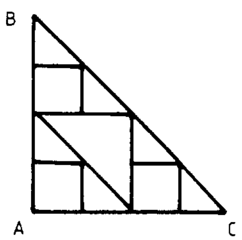

Figure 1. Second lattice approximant to the full Sierpinski gasket.

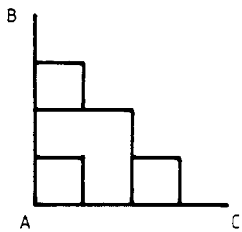

Figure 2. Second lattice approximant to a depleted Sierpinski gasket.

Although real-space methods are available for the same class of problems (Given and Mandelbrot 1984), the method best suited to showing universality is found in the fractal analogue to the momentum-space renormalisation group of Wilson *et al* as described in GM. In the language of GM, any two fractal lattices with the same 'tie generating scheme' have the same conductivity and diffusing scaling behaviour. Thus, for the lattice of figure 1, $\rho = \frac{5}{3}$ and $\alpha = 5$, where ρ is the asymptotic ratio of resistances measured at successive stages of iteration, that is

$$\rho = \lim_{n\to\infty} (R_{n+1}/R_n) = \lim_{n\to\infty} (\bar{R}_{n+1}/\bar{R}_n).$$

On the other hand, HBA claims $\rho = \frac{7}{4}$ and $\alpha = \frac{21}{4}$.

Since the lattices in figures 1 and 2 have only three external contact points, each can be modelled to any order of decoration via star–triangle transform by a triangle with two effective resistors: R between either A and B or A and C, and $\bar{R}$ between B and C. Furthermore, let R' (resp. R'') denote the two point resistances between either A and B or A and C (resp., B and C). The renormalisation group for these quantities has a single stable fixed point, satisfying $\lim_{n\to\infty} (\bar{R}_n/R_n) = 1$. The full Sierpinski gasket (figure 1) satisfies $(\bar{R}_1/R_1) = 1$, which means that it attains scaling behaviour immediately. The same is true of all the full Sierpinski gaskets irrespective of their base. The conductivity calculation by HBA tacitly assumes that this simple scaling behaviour also applies to other lattices, but this is not so. In fact, the depleted Sierpinski gasket in figure 2 clearly satisfies $(\bar{R}_1/R_1) = \infty$, and in this sense is the most slowly convergent member of its universality class. The values 1 and $\frac{7}{4}$ are those of the two-point resistance R' for the first and second iteration of the renormalisation

Table 1. Ratio of successive values of R, $\bar{R}$, R' and R'', as obtained by iterating the renormalisation group of Given and Mandelbrot (1983).

Stages	R	$\bar{R}$	R'	R''
2/1	2.000 000	0	1.750 000	1.500 000
3/2	1.875 000	1.000 000	1.730 769	1.538 462
4/3	1.807 692	1.292 801	1.716 216	1.567 568
5/4	1.766 820	1.426 029	1.705 174	1.589 653
6/5	1.740 065	1.500 176	1.696 743	1.606 514
7/6	1.721 640	1.546 410	1.690 262	1.619 476
8/7	1.708 487	1.577 358	1.685 246	1.629 508
9/8	1.698 850	1.599 074	1.681 341	1.637 317
10/9	1.691 647	1.614 819	1.678 286	1.643 428
11/10	1.686 184	1.626 507	1.675 886	1.648 229
12/11	1.681 993	1.635 332	1.673 993	1.652 014
13/12	1.678 750	1.642 084	1.672 497	1.655 006
14/13	1.676 221	1.647 299	1.671 311	1.657 378
15/14	1.674 240	1.651 359	1.670 369	1.659 261
16/15	1.672 682	1.654 536	1.669 621	1.660 758
17/16	1.671 451	1.657 036	1.669 025	1.661 950
18/17	1.670 476	1.659 008	1.668 550	1.662 900
19/18	1.669 703	1.660 569	1.668 171	1.663 658
20/19	1.669 089	1.661 807	1.667 869	1.664 262
21/20	1.668 600	1.662 791	1.667 628	1.664 745
22/21	1.668 210	1.663 573	1.667 435	1.665 130

group. However, as mentioned, these values fail to yield its scaling behaviour. Table 1 shows that even after six iterations, the ratio of successive values of R is 1.740 065, which is barely below $\frac{7}{4}$. The asymptotic value is not approached until the 20th generation. Thus, it is not surprising that the moderately large MC simulation reported in HBA should have yielded a poor value for ρ and hence should have seemed to confirm their incorrect analysis. Table 2 confirms that the renormalisation used to obtain table 1 is exact.

Table 2. Exact values of R, $\bar{R}$, R' and R'', as obtained by using Kirchhoff's laws. The ratios of successive lines in table 2 are identical to the values in table 1.

Stages	R	$\bar{R}$	R'	R''
1	1	∞	1	2
2	2	12	$\frac{7}{4}$	3
3	$\frac{15}{4}$	12	$\frac{315}{104}$	$\frac{60}{13}$
4	$\frac{26085}{3848}$	$\frac{163560}{10543}$	$\frac{40005}{7696}$	$\frac{3480}{481}$

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